

Fractional Langevin equation to describe anomalous diffusion.

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Abstract

A Langevin equation with a special type of additive random source is considered. This random force presents a fractional order derivative of white noise, and leads to a power-law time behavior of the mean square displacement of a particle, with the power exponent being noninteger. More general equation containing fractional time differential operators instead of usual ones is also proposed to describe anomalous diffusion processes. Such equation can be regarded as corresponding to systems with incomplete Hamiltonian chaos, and depending on the type of the relationship between the speed and coordinate of a particle yields either usual or fractional long-time behavior of diffusion. Correlations with the fractional Fokker-Planck equation are analyzed. Possible applications of the proposed equation beside anomalous diffusion itself are discussed.

1 Introduction

In recent years, growing attention has been focused on the processes that take place in random disordered media, and in dynamic systems demonstrating chaotic behavior. A special place here is taken by the systems with incomplete Hamiltonian chaos, which trajectories in the phase space can be portrayed as a set of "islands around islands" with self-similar (fractal) structure. In such systems the lifetime of any state of regular motion is also random. Incomplete chaos results in a few interesting phenomena that, among others, include anomalous properties of transport processes. That is associated with the fact that the islands of stability in this case become to act as a system of traps with a certain given distribution of the trapping time. For example, well known is the phenomenon of anomalous diffusion that is characterized by the time function of the mean squared displacement of a diffusing particle, which is described not by the Einstein's law but by a power function with fractional exponent [1, 2].

$$\langle(\Delta x)^2\rangle \propto t^\alpha \quad \alpha \neq 1$$

In most cases, such behavior is considered to be connected with self-similar properties of the diffusion medium. As this takes place, the Fokker-Planck equation describing such diffusion process was shown to involve the integrodifferential Riemann-Liouville operators I^ν and D^ν of fractional order [3] (see Appendix A). Possible interpretation of the physical meaning of these operators was offered in [4, 5], and assumes that a system described by equations with fractional derivatives or integrals possesses a "selective" memory that acts only in the points within a set of dimensionality ν , and is in accordance with the ideas about a certain self-similar (say, Levy-type or fractal) distribution of traps and waiting times [5]. Fractional Fokker-Planck equations were studied in [5]-[9], and proved to be quite useful to model anomalous diffusion processes. The most general form of such equations was introduced in [5, 8, 9] and in a simple case looks

$$\frac{\partial^\nu}{\partial t^\nu} n = D \frac{\partial^{2\gamma}}{\partial x^{2\gamma}} n \quad (1)$$

where n is the concentration of diffusing substance and the fractional differential operators on the right-hand side and on the left-hand side are either fractional Riemann-Liouville derivatives [5]-[8] or so called local fractal derivatives [9].

Another method of the anomalous transport properties description refers to their microscopic motion and involves introducing fractional derivatives into a stochastic process that characterizes the Brownian motion. As the starting point to be changed the Wiener stochastic process that defines the displacement of a particle with time is taken

$$x - x_0 = \int_0^t F(t') dt'$$

where $F(t)$ is the Gaussian white noise (the brackets here mean averaging over the possible realizations of the process),

$$\langle F(t) \rangle = 0, \quad \langle F(t_1)F(t_2) \rangle = q\delta(t_1 - t_2)$$

and the fractal Brownian motion is obtained by substituting the Riemann integral for a fractional one [10]-[13]

$$x - x_0 = I^\alpha F(t) \equiv \frac{1}{\Gamma(\alpha)} \int_0^t \frac{F(t')}{(t - t')^{1-\alpha}} dt' \quad (2)$$

The fractional integral of the white noise was named the fractional noise. This noise is Gaussian but nonstationary. Its correlation function has the form

$$\begin{aligned} \langle x(t_1)x(t_2) \rangle &= \frac{t_>^{\alpha-1} t_<^\alpha}{\alpha [\Gamma(\alpha)]^2} {}_2F_1(1, 1 - \alpha; 1 + \alpha; t_</t_>) \\ t_< &= \min(t_1, t_2), \quad t_> = \max(t_1, t_2) \\ \langle x(t)x(t) \rangle &\propto t^{2\alpha-1} \end{aligned} \quad (3)$$

In principle, this noise can be made stationary, but only provided that $-\infty$ is taken in Eq.(2) as the lower limit of integration instead of 0. However, in this case one has to somewhat modify the power kernel in order to provide the convergence of the integral. We will briefly discuss this below, at the end of the Section 2.

Fractal Brownian motion is of great interest not only from mathematics and theory of stochastic processes points of view but also in terms of physical applications. It can be used in describing polymer chains, electric transport

in disordered semiconductors, diffusion on comb-like structures, and so on. However, it is well known that in many cases the most convenient way of describing the Brownian diffusion of particles is not the Wiener process but rather the Ornstein-Uhlenbeck process (or Langevin method [2, 14, 15]) that is based on the solution of stochastic differential equation

$$\frac{d}{dt}v = -\gamma v + F(t) \quad (4)$$

where v is the velocity of a Brownian particle, γ means the factor of liquid friction, and $F(t)$ is the random source characterizing the properties of medium where diffusion occurs. Displacement of a particle and its path are determined not directly but through the integration of the instant velocities. Apart from the problem of Brownian motion itself, this method is widely applied to describe various systems subjected to external noise. This brings up the question of whether the Langevin equation can be written for anomalous diffusion as well, and if so, what will be the structure of the corresponding random source in it. In principle, if the Fokker-Planck equation is known then the Langevin equation can be derived from it. However, this method involves certain difficulties, and gets even more complicated because of the intricate structure of fractional integrodifferential operators. So, we shall postulate the form of such equation in further sections of our paper. Actually, we shall consider three forms of such equation that have in common the method of introducing the memory by means fractional derivatives yet differ in the properties of medium and in the ways of setting up the problem.

2 Langevin Equation with Fractional Derivatives

a. As it has been mentioned above, introducing the fractional differential operators into the Fokker-Planck equation makes it possible to describe the anomalous transport process quite correctly. Therefore, let us consider the following equation that differs from the usual Langevin equation by replacing the first derivative with respect to time by the fractional derivative of order ν

$$\frac{d^\nu}{dt^\nu}v = -\gamma v + F(t) \quad (5)$$

Applying the fractional integral operator to both left-hand and right-hand sides of this equation, one has

$$v - v_0 =_0 I_t^\nu(-\gamma v + F(t)) \quad (6)$$

Expressing the fractional integral in the explicit form, we rewrite it as follows

$$v = v_0 + A(t) - \frac{\gamma}{\Gamma(\nu)} \int_0^x \frac{v(x')}{(x-x')^{1-\nu}} dx', \quad A(t) = \frac{1}{\Gamma(\nu)} \int_0^x \frac{F(x')}{(x-x')^{1-\nu}} dx' \quad (7)$$

This equation can be easily solved by standard techniques for Volterra integral equations, and its solution has the form

$$v = v_0 E_{1,\nu}(-\gamma t^\nu) + \int_0^t F(t')(t-t')^{\nu-1} E_{\nu,\nu}[-\gamma(t-t')^\nu] dt' \quad (8)$$

where $E_{\alpha,\beta}(z)$ is so called Mittag-Leffleur function [16]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha + \beta k)}$$

If $\nu = 1$ then Eq.(8) reduces to the solution of the usual Langevin equation [14]

$$v = v_0 e^{-\gamma t} + \int_0^t F(t') e^{-\gamma(t-t')} dt'$$

If $F(t)$ is taken, as usual, to be a Gaussian δ -correlated source with zero mean, then the velocity correlation function has the form

$$\begin{aligned} \langle v(t_1)v(t_2) \rangle &= v_0^2 E_{1,\nu}(-\gamma t_1^\nu) E_{1,\nu}(-\gamma t_2^\nu) \\ &+ q \int_0^{\min(t_1,t_2)} dt' \frac{E_{\nu,\nu}[-\gamma(t_1-t')^\nu]}{(t_1-t')^{1-\nu}} \frac{E_{\nu,\nu}[-\gamma(t_2-t')^\nu]}{(t_2-t')^{1-\nu}} \end{aligned} \quad (9)$$

The integral from the right-hand side can be taken in the explicit form only if we introduce $t_>$ and $t_<$ from Eq.(3), which is rather inconvenient. However, we can obtain several important results without having to calculate

the integral directly. It is only essential that the integral is a symmetrical function of its arguments t_1 and t_2 . Further, let us find the corresponding mean squared displacement of the particle, which motion is described by the equation Eq.(5). It follows the expression

$$\langle(\Delta x)^2\rangle = \int_0^t \int_0^t \langle v(t_1)v(t_2)\rangle dt_1 dt_2 \quad (10)$$

which, as shown in the Appendix B, for correlation function of Eq.(9) can be reduced to

$$\begin{aligned} \langle(\Delta x)^2\rangle &= v_0^2 \left[t E_{2,\nu}(-\gamma t^\nu) \right]^2 \\ &+ \frac{q}{\gamma^2} \left(t - 2t E_{2,\nu}(-\gamma t^\nu) + \int_0^t \left[E_{1,\nu}(-\gamma t^\nu) \right]^2 dt \right) \end{aligned} \quad (11)$$

At large values of the argument $E_{1,\nu}(-\gamma t^\nu) \propto E_{2,\nu}(-\gamma t^\nu) \propto 1/\gamma t^\nu$. Therefore, the integral in Eq.(11) necessarily converges at $t \rightarrow \infty$ for $1/2 < \nu$, and the leading term in the asymptotics is

$$\langle(\Delta x)^2\rangle = \frac{q}{\gamma^2} t \quad (12)$$

just as it is the case for the classical Langevin equation with the first derivative with respect to time. Even when $\nu < 1/2$ and the integral diverges, it increases more slowly than the first term and the linear asymptotics remains. Leaving the discussion of this point for later, we shall try to modify somehow the initial equation.

b. First, it must be noted that in Eq.(11) we assumed that, as usually, the velocity v was defined as the first derivative of the coordinate with respect to time and, therefore, $x = \int_0^t v(t)dt$. Actually, it makes sense to consider a more general relationship

$$x = \frac{1}{\Gamma(\nu)} \int_0^t \frac{v(t')}{(t-t')^{1-\nu}} dt' \quad (13)$$

that corresponds to the most complete possible description of the system memory by means of the fractional integrals - the particle's displacement is

defined by velocity only in the points within a time interval of dimension ν . Though in this case the physical meaning of the corresponding inverse definition of velocity as a fractional derivative of the coordinate with respect to time needs to be explained. In order to clear this point, let us recollect that microscopic motion of a diffusing particle represents a twisted and everywhere nondifferentiable curve. For such curve, however, one can often find a derivative of fractional order [19] that, in fact, is a derivative of the trajectory, averaged with a power weight, and the observed motion of the particle is the thus averaged motion. In terms of memory, it means that for fractal paths (which are even the paths of classical Brownian particle) some of the instant velocities and displacements do not contribute into the resulting macroscopic motion. As this occurs, the behavior of the solution changes, and diffusion becomes anomalous. In this case instead of Eq.(11) we obtain

$$\begin{aligned} \langle(\Delta x)^2\rangle &= {}_0I_{t_1}^\nu {}_0I_{t_2}^\nu \langle v(t_1)v(t_2)\rangle = v_0^2 \frac{\left[E_{1,\nu}(-\gamma t^\nu) - 1\right]^2}{\gamma^2} \\ &+ \frac{2qt^{2\nu-1}}{\gamma^2} \sum_{k,l=1}^{\infty} \frac{(-\gamma)^{k+l}}{\Gamma(\nu + \nu k)\Gamma(\nu l)} \frac{\Gamma(\nu k + \nu l + \nu - 1)}{\Gamma(\nu k + \nu l + 2\nu)} t^{\nu k + \nu l} \end{aligned} \quad (14)$$

At large times the first term tends to v_0^2/γ^2 , and the series in the second term converges (see Appendix C). Therefore we finally have

$$\langle(\Delta x)^2\rangle \propto N \frac{q}{\gamma^2} t^{2\nu-1}, \quad 1 < N < \frac{2}{\Gamma(\nu)} \quad (15)$$

that coincides with the asymptotics obtained in [8, 9] for the generalized equation of fractal diffusion Eq.(1) if $\gamma = 1$, and with the asymptotics of fractional Brownian motion [10]-[13]. The larger ν the faster the particle moves. More accurate estimations of factor N can be obtained either following the method similar to that stated in the Appendices B and C (which in this case would be quite a challenge) or numerically.

c. Let us now investigate separately the influence of memory and fractal behavior on the regular and random components of the force acting on a particle. It turns out that the same fractal asymptotic behavior can be demonstrated by taking into account only the memory for the random force component in the initial Langevin equation, which has a more clear physical

meaning (cf. Eq.(7)):

$$v = v_0 - \gamma \int_0^t v(t') dt' + \frac{1}{\Gamma(\nu)} \int_0^t \frac{F(t')}{(t-t')^\nu} dt' \quad (16)$$

In fact, it means that in Eq.(4) the random source is not δ -correlated but represents the fractional derivative of the δ -correlated process $g(t)$, which can be taken in the sense of the generalized functions [18]

$$F(t) = {}_0D_t^{1-\nu} g(t) \quad \langle g(t_1)g(t_2) \rangle = q\delta(t_1 - t_2) \quad (17)$$

Similar equation was actually considered in [20]. It was shown numerically there that when in the Langevin equation Gaussian white noise is replaced by the fractional Gaussian noise [10, 11] it can yield the spectra for homogeneous Eulerian and Lagrangian turbulence. However, equation Eq.(16) can be easily solved analytically, and the solution reads

$$v = v_0 e^{-\gamma t} + \int_0^t F(t')(t-t')^{\nu-1} E_{\nu,1}[-\gamma(t-t')] dt' \quad (18)$$

The corresponding expression for correlation is

$$\begin{aligned} \langle v(t_1)v(t_2) \rangle &= v_0^2 e^{-\gamma(t_1+t_2)} \\ &+ q \int_0^{\min(t_1,t_2)} dt' \frac{E_{\nu,1}[-\gamma(t_1-t')]}{(t_1-t')^{1-\nu}} \frac{E_{\nu,1}[-\gamma(t_2-t')]}{(t_2-t')^{1-\nu}} \end{aligned} \quad (19)$$

Calculation of the mean squared displacement is similar to that in the Appendix B and results in

$$\begin{aligned} \langle (\Delta x)^2 \rangle &= v_0^2 \frac{(1 - e^{-\gamma t})^2}{\gamma^2} + \frac{q}{\gamma^2} S \\ S &= 2 \sum_{k,l=0}^{\infty} \frac{(-\gamma)^{k+l}}{\Gamma(1+\nu+k)\Gamma(\nu+l)} \frac{t^{k+l+2\nu+1}}{(k+l+2\nu)(k+l+2\nu+1)} \end{aligned} \quad (20)$$

By differentiating the sum two times, we obtain

$$\frac{d^2}{dt^2} S = \frac{d}{dt} \left[t^\nu E_{1+\nu,\nu}(-\gamma t) - 1 \right]^2$$

from which it follows that when $t \rightarrow \infty$ it is the second term that defines the asymptotic behavior of the displacement

$$\langle(\Delta x)^2\rangle \propto \frac{1}{(2\nu-1)[\Gamma(\nu+1)]^2} \frac{q}{\gamma^2} t^{2\nu-1} \quad (21)$$

which agrees with Eq.(15) up to the constant multiplier.

d. Finally, let us consider the third case, that is when the memory is taken into account only for the friction force

$$v = v_0 - \gamma \frac{1}{\Gamma(\nu)} \int_0^t \frac{v(t')}{(t-t')^\nu} dt' + \int_0^t v(t') dt' \quad (22)$$

where $F(t)$ is again the Gaussian white noise (cf. Eqs.(7), (16)). It means that now it is the dissipative force that is proportional to the fractional derivative of velocity $f_{diss} = {}_0D_t^{1-\nu}v(t)$. Its solution is

$$v = v_0 E_{1,\nu}(-\gamma t^\nu) + \int_0^t F(t') E_{\nu,\nu}[-\gamma(t-t')^\nu] dt' \quad (23)$$

The contribution from the second term in the mean squared displacement results in the power function of time in the following form

$$\langle(\Delta x)^2\rangle \propto M \frac{q}{\gamma^2} t^{3-2\nu}, \quad 1 < M < \frac{1}{\Gamma(1+\nu)} \quad (24)$$

It should be noted here that, unlike the previous two cases, here no limitations are imposed on the value of ν , and depending on whether it is greater or less than 1, we can obtain either subdiffusion with the power exponent less than 1, or superdiffusive behavior with the power exponent greater than 1, observed, for instance, for a phase for wave propagation in nonlinear or random media. The greater is ν , the slower the particle moves (it was the other way around in the previous cases), because at $\nu > 1$ the corresponding dissipative force is no longer a fractional derivative of the velocity but a fractional integral, and, hence, its influence increases. At $\nu > 3/2$, the mean squared displacement Eq.(24) tends to zero with time increasing, which corresponds to a stop in the particle motion since the terms neglected in Eq.(24), as well as in Eqs.(14)-(15), are either vanishingly small or constant,

and means that particle energy dissipates faster than it is pumped. On the other hand, limitations $1/2 < \nu < 1$ were used in the previous cases only to simplify the estimations for the sums which were shown to be convergent and therefore in principle it must not affect the main results and they also can give sub- and superdiffusion behavior. But because of the fact that prehistory influences only on the dissipative force, now diffusion is impeded when we increase the value of ν .

Let us now analyze the results obtained. If we substitute the first derivative in the Langevin equation for the fractional one but use the same relationship for velocity and coordinate, then the solution possesses the same linear asymptotic behavior as the initial solution does. This means that taking into account the memory for the friction force and random force at the same moments of time does not affect the particle's motion at large times but only in the beginning of the motion. In a sense, this is quite reasonable, since a random source provides a particle with an additional energy, and friction results in its dissipation. At larger times, when the system arrives at the stable state, these two processes compensate each other, but only provided that they have the same duration.

Using the relationship Eq.(13) means that the particle's motion equation written in the Newtonian form reads

$$\frac{d^{2\nu}x}{dt^{2\nu}} = \frac{d^\nu x}{dt^\nu} + F(t) \quad (25)$$

and is the most general in the sense of memory calculation - every derivative with respect to time becomes fractional. Then anomalous properties of the diffusion arise from fractional integration of velocity, that is, from the assumption only part of instant velocities contribute to the final path.

The anomalous diffusion is also the result of the Langevin equation with a source that is a fractional derivative of the white noise. The autocorrelation function of such noise, just like that in Eq.(2), has a power behavior yet negative values of the exponent and therefore appears to be a generalization of the flicker-noise (see [2]). It should be noted that a stationary noise can be obtained only by using the Liouville fractional derivative with the infinite lower limit. However, when used directly in the equations Eqs.(7), (16), (22), these derivatives would lead to the divergences because of the power integrands. Besides, this is not quite reasonable, since in real situations there is always a moment at which the motion starts, and assuming this moment as infinitely distant is possible only in a system where characteristic relaxation

scales exist. Fractional Brownian motion, however, can not be assigned to such processes - its memory is described by a power function and leads to the long-range correlations having no time scale of their own. Nevertheless, the nonstationary of the noise should not lead to a misunderstanding. As we shall illustrate below, such process can be regarded as stationary only in a wide sense (the corresponding transition probabilities depend uniquely upon the duration of the transition Δt) only for small Δt , that is for $\Delta t \ll t$.

The absence of stationarity is an important and rather obvious property of anomalous diffusion, although it is not always given proper consideration. Here it is relevant to note the following. Normally, the Brownian motion can be described with two different stochastic processes. The Ornstein-Uhlenbeck process is strictly stationary but does not have independent increments. Moreover, its increments are not even uncorrelated. The Wiener process, which is the integrated Ornstein-Uhlenbeck process in the limit of intense friction and noise, has stationary independent increments, but is neither strictly stationary nor a wide-sense stationary. The process under investigation with a fractional derivative of the white noise seems to be an intermediate process that does not have stationary increments but is asymptotically stationary in a wide sense. The fractal nature of motion leads to the fact that even this kind of stationarity is observed only within short periods of time, which means that the process becomes quasistationary. This conclusion is in a good agreement with the notion that anomalous diffusion is just an intermediate asymptotic behavior for systems of certain types.

Finally, if the prehistory affects only the dissipative force acting on a particle, its behavior also becomes anomalous.

3 Probability distributions

Let us now study the probability distributions that arise from the stochastic processes described above. For the sake of convenience, let us consider only the process with the noise that is a fractional derivative of the white noise Eq.(16). The other cases are considered in perfect analogy to this one and give similar results.

The distribution of the particle coordinate will be Gaussian. It follows from the fact that the equations are linear both with respect to the external additive noise and the stochastic variable. In order to determine the transition probability $P(x, t + \Delta t; y, t)$ let us consider transition moments at large

times

$$M_n = |x - y|^n = \int_t^{t+\Delta t} \dots \int_t^{t+\Delta t} \langle v(t_1) \dots v(t_n) \rangle dt_1 \dots dt_n = \begin{cases} 0, & n = 2k + 1 \\ \frac{(2k)!}{2^k k!} I^k, & n = 2k \end{cases} \quad (26)$$

where

$$\begin{aligned} I &= \int_t^{t+\Delta t} \int_t^{t+\Delta t} \langle v(t_1) v(t_2) \rangle dt_1 dt_2 \\ &= \int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 \int_0^{\min(t_1, t_2)} \frac{E_{\nu,1}[-\gamma(t_1 - t')]}{(t_1 - t')^{1-\nu}} \frac{E_{\nu,1}[-\gamma(t_2 - t')]}{(t_2 - t')^{1-\nu}} dt' \end{aligned} \quad (27)$$

This implies that the probability of transition obeys the Gaussian distribution with dispersion I . In order to identify time dependence of the dispersion, let us rewrite the internal integral in the form $\int_0^{t+\Delta t} = \int_0^t + \int_t^{t+\Delta t}$. In this case Eq.(27) will have two terms, the first of which allows changing the order of integration since the limits of integration are no longer dependent on the intrinsic variables:

$$I_2 = \int_0^t dt' \left[\int_0^{t+\Delta t} \frac{E_{\nu,1}[-\gamma(t_1 - t')]}{(t_1 - t')^{1-\nu}} dt_1 \right]^2$$

At $\Delta t \ll t$ we then get

$$I_2 = A \frac{q}{\gamma^2} \frac{\Delta t^2}{t^{3-2\nu}}, \quad \frac{1}{2\nu [\Gamma(\nu)]^2} < A < \frac{1}{(2\nu - 1) [\Gamma(\nu)]^2} \quad (28)$$

The second terms that appears in Eq.(27) after the substitution $t_1 \rightarrow t_1 - t$, $t_2 \rightarrow t_2 - t$, $t' \rightarrow t' - t$ reduces to the mean squared displacement given by Eq.(21) during time Δt

$$I_1 = \frac{1}{(2\nu - 1) [\Gamma(\nu + 1)]^2} \frac{q}{\gamma^2} t^{2\nu - 1}$$

which is much greater than I_2 if $\Delta t \ll t$. Hence,

$$P(x, t + \Delta t; y, t) = \sqrt{\frac{\gamma^2(2\nu - 1)[\Gamma(\nu + 1)]^2}{2\pi q} \frac{1}{\Delta t^{2\nu-1}}} \times \exp\left(-\frac{\gamma^2(2\nu - 1)[\Gamma(\nu + 1)]^2 |x - y|^2}{2q \Delta t^{2\nu-1}}\right) \quad (29)$$

This expression also gives a "suitable" result for the coordinate distribution $W(x, t)$ of the wandering particle that agrees with [2, 11, 10, 13]

$$\begin{aligned} W(x, t) &\equiv P(x, t; 0, 0) = \frac{1}{\sqrt{2\pi B t^{2\nu-1}}} \exp\left(-\frac{x^2}{2B t^{2\nu-1}}\right) \quad (30) \\ B &= \frac{1}{(2\nu - 1)[\Gamma(\nu + 1)]^2} \frac{q}{\gamma^2} \end{aligned}$$

and follows from the fact that if we take $y = 0, t_0 = 0$ as the initial point of transition, then the $I_2 = 0$ holds disregarding the value of Δt . Therefore, at large times the described process becomes quasistationary in a wide sense, that is, the probability of transition during a small time interval Δt depends on the value of Δt only. However, if the condition $\Delta t \ll t$ is not fulfilled, then I_2 can no longer be given with the simple expression like Eq.(28) and thus can not be neglected as compared to I_1 . It should also be noted that such process on any time scales is not Markovian since the equations Eqs.(29)-(30) do not comply with the Chapman-Kolmogorov equation

$$P(x, t + \Delta t) = \int P(x, t + \Delta t; y, t) P(y, t) dy$$

which can be verified by direct substitution. Indeed, if we introduce in Eq.(30) new time variable as $\tau = t^{2\nu-1}$, then the Markovian Gaussian process will be that with the transition probability distribution

$$P(x, \tau + \Delta\tau; y, \tau) = \frac{1}{\sqrt{2\pi D \Delta\tau^{2\nu-1}}} \exp\left(-\frac{|x - y|^2}{2D \Delta\tau^{2\nu-1}}\right) \Delta\tau = (t + \Delta t)^{2\nu-1} - t^{2\nu-1}$$

and at small Δt

$$\Delta\tau \approx (2\nu - 1)t^{2\nu-2} \Delta t$$

It is worth to note that in this case the property of wide- sense stationarity is lost for all time scale.

Non-Markovity is a consequence of the memory of the past process existence. If for a Markovian process the future is uniquely determined by the present, then in the initial equation Eq.(16) the behavior of a particle in the next moment of time is, generally, dependent on the whole previous history starting from the very beginning of the motion, and the transition probability $P(x, t + \Delta t; y, t)$ does not depend on the time t only when $\Delta t \rightarrow 0$. In this connection the question of transition from Eq.(16) to the Fokker-Planck-type equation still remains to be solved. Indeed, the proposed in [5, 9] method is based on the fractional Taylor series expansion of the transition probabilities in the Chapman-Kolmogorov equation. For example, according to [5], the fractional Fokker-Planck equation for the transition probability given by Eq.(29) takes the form

$$\frac{\partial^{2\nu-1}}{\partial t^{2\nu-1}} W(x, t) = D \frac{\partial^2}{\partial x^2} W(x, t) \quad (31)$$

Solution of this equation was obtained in [8, 21] and shown to be

$$W_1(x, t) = \frac{1}{\sqrt{Dt^{2\nu-1}}} H_{1,1}^{1,0} \left(\frac{x^2}{Dt^{2\nu-1}} \mid \begin{matrix} (1, \nu - 1/2) \\ (1, 1) \end{matrix} \right) \quad (32)$$

where $H_{1,1}^{1,0}$ is the so-called Fox function [17, 22]. However, according to Eq.(30)

$$W(x, t) = \frac{1}{\sqrt{2\pi Dt^{2\nu-1}}} \exp \left(-\frac{x^2}{Dt^{2\nu-1}} \right) = H_{0,1}^{1,0} \left(\frac{x^2}{Dt^{2\nu-1}} \mid \begin{matrix} - \\ (1, 1) \end{matrix} \right) \neq W_1(x, t) \quad (33)$$

Therefore, the solution Eq.(33) does not satisfy Eq.(31) though the time and the coordinate of a particle appears in its solution in the "correct" combination $x^2/t^{2\nu-1}$. However, as the stochastic process under consideration is not globally stationary, this fact seems to influence the structure of the corresponding Fokker-Planck equation. In particular, such equation may also include the terms that characterize the sources and drains of probability, and therefore its structure will no longer be as simple as in Eq.(31). Probably this problem may be solved by studying the waiting time probabilities and then writing down the equations similar to those used in [7].

The average kinetic energy of a particle can be obtained as

$$\langle E \rangle = \langle [v(t)]^2 \rangle \quad (34)$$

Substituting here the expression for the velocity correlation Eq.(19) and taking into account that at large times, when the motion becomes stable, the particle "forgets" the initial velocity and we can neglect the first term in it (see also Eq.(26)), yields

$$\langle E \rangle = \frac{mq}{2} \int_0^t \tau^{2\nu-2} [E_{\nu,1}(-\gamma\tau)]^2 dx \quad (35)$$

Since when $t \rightarrow \infty$ $E_{\nu,1}(-\gamma\tau) \propto 1/\gamma\tau$ and in the vicinity of zero the Mittag-Leffleur function is bounded, for $1/2 < \nu$ the integral in Eq.(35) converges. Therefore (see [14], the velocity of the moving particle will obey the Maxwellian distribution.

Probability distribution for the processes described in the paragraphs **2b** (Eqs.(7) and (13)) and **2d** (Eq. (22)) are obtained in the same way. Velocity distribution obeys the Maxwellian law, and the coordinate distribution $W(x, t)$ and transition probability $P(x, t + \Delta t; y, t)$ at large times t and small Δt follow the Gaussian law with the dispersion determined by the mean squared displacement over times t and Δt , respectively.

4 Conclusion

To conclude, we would like to note the following. The equations and models proposed in this paper are quite general and, represent a way to introduce and describe a certain class of Gaussian non-Markovian stochastic processes. Processes of this type are featured with the "selective" memory acting only in the moments of time distributed over a Cantor-type fractal set and taken into account by means of fractional derivatives, and show promise in describing stochastic processes in fractal media and systems with incomplete Hamiltonian chaos. One of such processes is anomalous diffusion, observed in a wide variety of systems. The method developed in this paper is an alternative to that utilizing the fractional Fokker-Planck equations, although we could not trace fully the relation between these two approaches and it still remains to be explored. Nevertheless, the two approaches give similar

results, and choosing between them should probably be determined by a type of problem to be solved and by practical convenience of calculations. Earlier the fractal Brownian motion was represented only as a Wiener stochastic process of fractional order (the integral of order ν from the white noise or, which is the same, the integral of the first order of the noise which is a fractional derivative of order $\nu - 1$ of the white noise). As shown in the present paper, the Ornstein-Uhlenbeck process with fractional noise leads to similar results and allows to obtain more general probability distributions in an easier way, as compared to the path integrals [13]. The equations proposed can also be easily generalized by adding linear and nonlinear terms, and be used for calculation of various statistic characteristics of real stochastic dynamic systems.

A Fractional derivatives

There are about two dozens of different definitions for fractional derivatives that are in one way or another adapted to various features of classes of functions for which they are defined. The most comprehensive description of the mathematical aspects of the issue is given in the monograph [3]. In physics the Riemann-Liouville fractional derivative is the most commonly used. Its definition goes back to the well-known Cauchy formula for multiple integrals

$${}_a I_t^n f(t) = \underbrace{\int_a^t dt \dots \int_a^t dt}_{n \text{ times}} f(t) = \frac{1}{(n-1)!} \int_a^t dt' \frac{f(t')}{(t-t')^{1-n}}$$

Substituting the factorial for the Euler gamma-function, we can generalize the formula by introducing the fractional exponent α as follows

$${}_a I_t^\alpha f(t) \equiv \frac{1}{\Gamma(\alpha)} \int_a^t dt' \frac{f(t')}{(t-t')^{1-\alpha}} \quad (\text{A.1})$$

This expression is referred to as the Riemann-Liouville fractional integral. The fractional derivative is then defined as an ordinary derivative of the integral of fractional order

$$\begin{aligned} {}_a D_t^\alpha f(t) &\equiv \frac{d^\alpha}{dt^\alpha} f(t) = \frac{d^n}{dt^n} {}_a I_t^{n-\alpha} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(t')}{(t-t')^{1+\alpha-n}} dt' \quad n-1 \leq \alpha < n \end{aligned} \quad (\text{A.2})$$

Fractional derivative can also be treated in the form of convolution with power function, and in this sense the definitions in Eqs.(A.1)-(A.2) can be easily transferred onto generalized functions.

B Calculation of the mean squared displacement

The mean squared displacement of a particle during time t is defined as

$$\langle(\Delta x)^2\rangle = \int_0^t \int_0^t \langle v(t_1)v(t_2) \rangle dt_1 dt_2 = v_0^2 \left[\int_0^t E_{1,\nu}(-\gamma t'^\nu) dt' \right]^2 + \frac{q}{\gamma^2} I \quad (\text{B.1})$$

where the integral in square brackets in the first term equals to $tE_{2,\nu}(-\gamma t^\nu)$, and the second terms takes the form (we expand the Mittag-Lieffleur function in a series)

$$\begin{aligned} I &= \sum_{k,l=1}^{\infty} \frac{(-\gamma)^{k+l}}{\Gamma(\nu k)\Gamma(\nu l)} \int_0^t dt_1 \int_0^t dt_2 \int_0^{\min(t_1,t_2)} (t_1 - t')^{\nu k - 1} (t_2 - t')^{\nu l - 1} dt' \\ &= \sum_{k,l=1}^{\infty} \frac{(-\gamma)^{k+l}}{\Gamma(\nu k)\Gamma(\nu l)} I_{kl} \end{aligned} \quad (\text{B.2})$$

Since the triple integral here does not change its form when interchanging $t_1 \leftrightarrow t_2$, then, assuming for certainty $t_1 < t_2$, we can write it as

$$I_{kl} = 2 \int_0^t dt_2 \int_0^{t_2} dt_1 \int_0^{t_1} dt' (t_1 - t')^{\nu k - 1} (t_2 - t')^{\nu l - 1}$$

This integral is now easy to calculate using the rules of fractional integration [3] and the properties of Gauss generalized hypergeometric functions ${}_2F_1$ [17]

$$\begin{aligned} I_{kl} &= 2 \int_0^t dt_2 \int_0^{t_2} dt_1 \Gamma(\nu k) {}_0I_{t_1}^{\nu k} (t_2 - t_1)^{\nu l - 1} \\ &= \frac{2\Gamma(\nu k)}{\Gamma(\nu k + 1)} \int_0^t dt_2 t_2^{\nu k + \nu l} \int_0^{t_2} \left(\frac{t_1}{t_2} \right)^{\nu k} {}_2F_1 \left(1, 1 - \nu l; 1 + \nu k; \frac{t_1}{t_2} \right) d\frac{t_1}{t_2} \\ &= \frac{2}{\nu k(\nu k + \nu l)} \int_0^t dt_2 t_2^{\nu k + \nu l} = \frac{2t_2^{\nu k + \nu l + 1}}{\nu k(\nu k + \nu l)(\nu k + \nu l + 1)} \end{aligned} \quad (\text{B.3})$$

Substituting in Eq.(11) yields

$$I_{kl} = 2t \sum_{k,l=1}^{\infty} \frac{(-\gamma t^{\nu})^{k+l}}{\Gamma(1+\nu k)\Gamma(\nu l)(\nu k + \nu l)(\nu k + \nu l + 1)} \quad (\text{B.4})$$

In order to evaluate the asymptotic behavior of this series at $t \rightarrow \infty$, let us note that

$$\begin{aligned} \frac{d^2 I}{dt^2} &= 2 \sum_{k,l=1}^{\infty} \frac{(-\gamma t^{\nu})^k}{\Gamma(1+\nu k)} \sum_{k,l=1}^{\infty} \frac{(-\gamma t^{\nu})^{l-1}}{\Gamma(\nu l)} \\ &= \frac{d}{dt} \left[\sum_{k=1}^{\infty} \frac{(-\gamma t)^{\nu k}}{\Gamma(1+\nu k)} \right]^2 = \frac{d}{dt} \left[(E_{1,\nu}(-\gamma t^{\nu}) - 1) \right]^2 \end{aligned} \quad (\text{B.5})$$

Integrated twice, and taken into account that $I(t=0) = \frac{dI}{dt}|_{t=0}$, for the mean squared displacement it gives

$$\begin{aligned} \langle (\Delta x)^2 \rangle &= v_0^2 \left[t E_{2,\nu}(-\gamma t^{\nu}) \right]^2 \\ &+ \frac{q}{\gamma^2} \left(t - 2t E_{2,\nu}(-\gamma t^{\nu}) + \int_0^t \left[E_{1,\nu}(-\gamma t^{\nu}) \right]^2 dt \right) \end{aligned} \quad (\text{B.6})$$

C The sum evaluation

We can find the asymptotic behavior of the double sum in Eq.(14) (we designate it as S_1) in the following way. First, let us find the upper boundary for this sum. To do this we substitute the second cofactor ($\nu > 1/2$, $k, l \geq 1$)

$$\begin{aligned} \frac{\Gamma(\nu k + \nu l + \nu - 1)}{\Gamma(\nu k + \nu l + 2\nu)} &= \frac{\Gamma(\nu k + \nu l + \nu)}{(\nu k + \nu l + \nu - 1)\Gamma(\nu k + \nu l + 2\nu)} \\ &< \frac{1}{\nu k + \nu l + \nu - 1} < \frac{1}{\nu l} \end{aligned} \quad (\text{C.1})$$

hence

$$S_1 < \sum_{k=1}^{\infty} \frac{(-\gamma t^{\nu})^k}{\Gamma(\nu + \nu k)} \sum_{l=1}^{\infty} \frac{(-\gamma t^{\nu})^l}{\Gamma(1 + \nu l)} = \left(E_{\nu,\nu}(-\gamma t^{\nu}) - \frac{1}{\Gamma(\nu)} \right) \left(E_{1,\nu}(-\gamma t^{\nu}) - 1 \right) \quad (\text{C.2})$$

Therefore, $S_1 < 1/\Gamma(\nu)$ when $t \rightarrow \infty$. In order to get the lower approximation for the sum (for $1/2 < \nu < 1$) we shall multiply it by t and differentiate twice. Reducing the argument in the gamma-function in the numerator by $\nu < 1$ and increasing the denominator by $(2 - 2\nu) < 1$ we then get

$$\begin{aligned} \frac{d^2}{dt^2}(S_1 t) &> \sum_{k,l=1}^{\infty} \frac{(-\gamma)^{k+l}}{\Gamma(1+\nu k)\Gamma(\nu l)} \frac{\Gamma(\nu k + \nu l - 1)}{\Gamma(\nu k + \nu l + 2)} t^{\nu k + \nu l} \\ &= \frac{1}{2} \frac{d}{dt} \left[\sum_{k=1}^{\infty} \frac{(-\gamma)^k t^{\nu k}}{\Gamma(1+\nu k)} \right]^2 \equiv \frac{1}{2} \frac{d}{dt} \left[E_{1,\nu}(-\gamma t^\nu) - 1 \right]^2 \quad (\text{C.3}) \end{aligned}$$

Since this inequality holds for every t , we can integrate the left-hand and right-hand sides, which yields $S_1 > 1/2$ for $t \rightarrow \infty$.

References

- [1] J.-P.Bouchaud, G.Georges, Phys.Rep., 1990, V.195, p.127
- [2] Yu.L.Klimontovich *Statistical theory of open systems* (Kluwer, Dordrecht, 1995)
- [3] S.G.Samko , A.A.Kilbas , O.I.Marichev, *Fractional integrals and derivatives - theory and applications* (Gordon and Breach, New York, 1993)
- [4] R.R.Nigmatullin, Theor. and Math. Phys., 1992, V.90, p.354 (in Russian)
- [5] G.M.Zaslavsky , Chaos, 1994, V.4, P.25-33
- [6] W.R.Schneider, W.Wyss, J. Math Phys. 1989, V.30, p.134
- [7] K.V.Chukbar, JETP (Sov. Zh. of Exp. and Theor. Phys), 1995, V.108, p.1875-1884
- [8] V.L.Kobelev, E.P.Romanov, L.Ya.Kobelev, Ya.L.Kobelev, Physics Doklady, 1998, Vol.43, No.8, p.487
- [9] K.M.Kolwankar, A.D.Gangal, Phys.Rev.Lett., 1998, V.80, P.214
- [10] B.Mandelbrot, J.W.Van Ness, 1968, SIAM Rev., V.10, p.422
- [11] E.Feder, *Fractals* (Plenum Press, New York, 1989).
- [12] W. Wyss, Found. of Phys. Lett., 1991, V.4 p.235
- [13] K.L.Sebastian, J. of Phys. A, 1995, V.28, p.4305
- [14] H.Risken, *The Fokker-Plank Equation.* (Springer, Berlin, 1984)
- [15] N.G. van Kampen, *Stochastic processes in physics and chemistry* (North-Holland, Amsterdam, 1981)
- [16] M.Dzhrbashyan, *Integral transformations and representation of functions in complex region* (Nauka, Moscow, 1976)(in Russian)
- [17] A.P.Prudnikov, Yu.A.Brychkov, O.I.Marichev, *Integrals and series. Part 3.* (Nauka, Moscow, 1986) (in Russian)

- [18] I.M.Gelfand, G.E.Shilov, *Generalized functions* (Academic Press, New York, 1964)
- [19] K.M.Kolwankar, A.D.Gangal, Chaos, 1996, V.6, P.505
- [20] Y.F.Shao, Physica D, 1995, V.83, P.461
- [21] I.Podlubny, pre-print UEF-02-94, Inst. of Exper. Phys. Slovak Acad. of Sci., 1994. (e-Print <http://xxx.lanl.gov/abs/funct-an/9710005>)
- [22] A.M.Mathai, R.K.Saxena, *The H-function with Applications in Statistics and Other Disciplines.* (New Delhi: Wiley, 1978)